Two generic frameworks for credit index volatility products and their application to credit index options

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Abstract

The extension of the single-name CDS option to the index case requires a careful analysis of the index "spread" –inlcuding the joint distribution of the index spread and the index loss. We first introduce an index spread that is closer to the single-name case, called *CDS-like* spread. We then compare it to the spread quoted in the market, in terms of forward, change of probability measure, treatment of convexity, etc. These frameworks are not sufficient to deal with the index loss in the option payoff. To cope with this, we use the *ad hoc* spread adjustment designed for the option by Pedersen [3] ; alternatively, we suggest to work conditionally on the spread to capture the loss distribution. Our methodologies can be used with any dynamics for the index spreads introduced, but the variety of these dynamics is not explored here –we essentially stick to the lognormal distribution as an example.

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1 Introduction

Increased liquidity for credit indices has enabled the development of index-based derivatives. Currently, the index option is the most, if not sole, liquid example. Its payoff is naturally expressed in terms of the index spread quoted on the market. But this index spread differs from a single-name CDS spread: the spread actually paid on an index contract is *not* the quoted spread, but a fixed contractual spread, along with an upfront payment that reflects the off-marketness of the trade. Therefore the quoted spread appears only as an intermediary tool to compute the upfront amount, and the toolbox for CDS spread modelling cannot be readily applied to the quoted index spread.

Therefore we start by introducing a new index spread, called *CDS-like spread*, which is a simple extension of the single-name case. A conversion formula allows an easy switch between the quoted and CDS-like spreads, and intuition is provided via numerical examples. In this new framework, we can extend to the index case the survival measure defined by Schönbucher for single-name options; cf. [6], [7] and [8] . Here the "numeraire" is the CDS-like index duration, which collapses to zero in case of an Armageddon event - a default of all names in the index basket, introduced by Brigo & Morini [1]. Still we can define a probability measure associated to this numeraire, which helps pricing payoffs involving the duration ; for other payoffs, we have to deal with convexity.

Then we jump to a direct modelling of the quoted index spread. Unfortunately the definition of the *forward* spread and duration proves difficult, both conceptually and in terms of practical implementation. We partly address this complexity with a couple of efficient approximations. From there we can follow the same route as in the CDS-like case : change of probability measure, and convexity treatment.

Having laid the foundations for index spread derivatives, we focus on the index option, and start by highlighting the hidden complexity of its payoff. In particular this payoff includes the cumulated loss upon exercise, which requires some joint modelling of the index spread and index loss. It is tempting to replace the random loss with its unconditional expectation, but this approximation proves very coarse in the case of stressed markets. A better alternative to the joint modelling is our loss-adjusted spread, that somehow incorporates the loss within the spread: this idea was initially suggested by Pedersen [3], and inspired further research [5], but our implementation is actually different. In [4], Jackson prices the option conditionally on the loss, but this approach requires to input a spread volatility for each loss level, as well as a loss distribution for the index , e.g. as implied by the market on index tranches: these dependencies are not desired. Instead we propose to build the loss distribution conditionally on the spread, using assumptions that are common when pricing CDO tranches. We are then left with a numerical integration over the spread distribution chosen.

2 Credit Index contract

2.1 Definition and notations

We remind that a credit index is simply a basket of p single-name CDS contracts, with a common trade date T_0 called the roll date, common quarterly payment dates $T_1, \ldots, T_n$ up to the maturity T_n, and a common contractual spread $S^c_{T_n}$.

Example 1 *The Series 8 of the iTraxx Crossover has a basket of* $p = 50$ *names; it was rolled on* $T_0 = 20$ *Sep* 2007, *and is available with the standard 3Y, 5Y, 7Y and 10Y maturities*[1]*, each with its own contractual spread. For example, the 5Y pays a contractual spread* $S^c_{5Y} = 375$ *bps.*

The (cumulated) index loss at time t is defined by:

$$L_t = \sum_{j=1}^{p} N^j \left(1 - R^j\right) \mathbf{1}_{\{\tau^j \leq t\}}$$

while the outstanding notional is simply

$$N_t = \sum_{j=1}^{p} N^j.\mathbf{1}_{\{\tau^j > t\}}$$

Here, for CDS number j,

- τ^j is the default time
- N^j is the notional (usually homogeneous - at least upon roll)
- R^j is the recovery rate

For the sake of simplicity, we will always assume deterministic interest rates, and an initial basket notional of 1: $\sum_{j=1}^{p} N^j = 1$.

2.2 Upfront amount Vs. Quoted spread

So as to enter an index contract at a given date t, an upfront payment U_{t,T_n} is made to reflect the off-marketness of an index contract struck at $S^c_{T_n}$. In practice, this upfront is communicated through a *quoted spread*[2] written S^q_{t,T_n}. Before we detail the relationship between this spread and the upfront, we need to introduce the *Flat* Risky Basis Point value, also known as Risky DVO1 (Discounted Value Of 1 basis point) or Risky Duration. We assume that $T_i \leq t < T_{i+1}$.

[1] For a roll date of 20 September 2007, these maturities are respectively: 20 December 2010, 2012, 2014 and 2017.

[2] This is not a "market spread" so to speak, as there is no index contract on the market paying the quoted spread. It is sometimes called "reference spread".

Definition 2 *The Flat RBP, written $F_{t,T_n}(S)$, is defined as the value at time t of a risky basis point paid between t and T_n, with the risk of a (virtual) flat spread curve with spread S and recovery 40%. Our convention is that the first payment will accrue from t to T_{i+1}.*

With a "relatively" flat interest rate curve, it is well-known that the default intensity curve resulting from a flat spread curve is almost[3] flat at $\frac{S}{1-R}$. When $t = T_i$, this leads to a simple approximation:

$$F_{t,T_n}(S) \simeq \sum_{k=i+1}^{n} (T_k - T_{k-1})\, ZC_{t,T_k} e^{-\frac{S}{1-R}(T_k - t)} \tag{1}$$

where $ZC_{t,T}$ is the risk-free zero-coupon at t for maturity T.

With these notations, the market convention is as follows: the protection buyer pays an upfront amount equal to:

$$U_{t,T_n} \triangleq \left(S^q_{t,T_n} - S^c_{T_n}\right) F_{t,T_n}\left(S^q_{t,T_n}\right) N_t = u_{t,T_n}\left(S^q_{t,T_n}\right) N_t \tag{2}$$

where the auxiliary function u_{T,T_n} is defined by:

$$u_{t,T_n}(s) \triangleq \left(s - S^c_{T_n}\right) F_{t,T_n}(s)\,. \tag{3}$$

At first glance, the use of the flat RBP may look artificial, but it actually mitigates operational risk:

- less data transfer: only one spread number is required to compute the upfront amount, while a usual RBP would require a full spread curve
- lower dependency on the CDS pricer of the counterparties: most pricing tools will coincide on the RBP value of a flat spread curve, so that the parties are likely to agree on the upfront amount

Note that the MtM (mark-to-market) at t of an existing index contract is precisely this amount U_{t,T_n}.

Remark 3 *For ease of notation, formula (2) omits the accrual of the* contractual *spread between T_i and t. This accrual actually reduces the upfront by an amount $S^c_{T_n}.(t - T_i)\, N_{T_i,t}$ where $N_{T_i,t}$ is some "averaged" index notional on the accrual period $[T_i, t]$. Note that this negligence is innocuous in the context of credit index options, given that the accrual term is netted in the final payoff.*

[3] This is actually exact if the spread is paid continuously, even with a steep interest rate curve.

2.3 A new tool: the CDS-like spread

As the index is simply a basket of single-name CDS contracts, we can write its RBP as a weighted sum:

$$RBP_{t,T_n} = \sum_{j=1}^{p} N^j \mathbf{1}_{\{\tau^j > t\}} RBP^j_{t,T_n} \tag{4}$$

The same applies for P_{t,T_n}, the present value of the index protection leg, also known as default leg or contingent leg. Given that the protection leg P^j_{t,T_n} for name j is equal to the spread leg, we can write

$$P^j_{t,T_n} = \mathbf{1}_{\{\tau^j > t\}} S^j_{t,T_n} RBP^j_{t,T_n},$$

where S^j_{t,T_n} is the market spread at time t for maturity T_n, and RBP^j_{t,T_n} is the value at time t of 1 risky bp paid between t and T_n. Finally:

$$P_{t,T_n} = \sum_{j=1}^{p} N^j \mathbf{1}_{\{\tau^j > t\}} S^j_{t,T_n} RBP^j_{t,T_n} \tag{5}$$

We can now introduce a *CDS-like spread* S_{t,T_n} for the index by setting:

$$S_{t,T_n} RBP_{t,T_n} \triangleq P_{t,T_n} \tag{6}$$

Remark 4 *Computing the CDS-like spread from (6) would require a preliminary work on the single-name spreads (rescaling to eliminate the index/single-name basis). But in practice, we will never use these single-name spreads themselves: whenever required, the CDS-like index spread will always be bootstrapped from the quoted index spreads: Appendix A details the conversion of quoted spread information into CDS-like information*

Inserting (4) and (5) in definition (6) shows the CDS-like index spread as a weighted average of the CDS spreads:

$$S_{t,T_n} = \frac{1}{\sum_{j=1}^{p} \omega^j} \sum_{j=1}^{p} \omega^j S^j_{t,T_n}$$

where the weights are:

$$\omega^j = N^j \mathbf{1}_{\{\tau^j > t\}} RBP^j_{t,T_n}$$

Now we can rewrite the upfront U_{t,T_n} in terms of the CDS-like spread and the index RBP[4]:

$$U_{t,T_n} = \left(S_{t,T_n} - S^c_{T_n}\right) RBP_{t,T_n} \tag{7}$$

[4] Note that the RBP term requires the term-structure $\left(S_{t,T_i}\right)_{i \le n}$ of the CDS-like spreads up to T_n, as opposed to the flat RBP which requires only the quoted spread for maturity T_n

2.4 Conversion formula

Equating (2) and (7), we link the two types of index spreads introduced:

$$\boxed{\left(S^q_{t,T_n} - S^c_{T_n}\right) F_{t,T_n}\left(S^q_{t,T_n}\right) N_t = \left(S_{t,T_n} - S^c_{T_n}\right) RBP_{t,T_n}} \tag{8}$$

Note that the index notional N_t does not appear on the right-hand side, given that the names that have defaulted before t are already excluded by the default indicators in equations (4) and (5).

Which of these two spreads spread is more useful ?

- Through the flat RBP, the quoted spread S^q_{t,T_n} actually provides integrated information in the time dimension, which makes comparison at different maturities more difficult than using CDS-like spreads. We could therefore be tempted to consider the quoted spread only as a tool to compute the upfront; nevertheless it is so close to the CDS-like spread (see appendix B) that it remains meaningful.
- We will see below that we can adapt the single-name toolbox for a use with CDS-like index spreads. Nevertheless, the CDS-like spread is *not* quoted, and no index derivative is likely to be naturally expressed in terms of this spread. This translates into complex pricing issues, as detailed in §3.1.

3 A toolbox for index spread derivatives

In this paper we do not focus on index derivatives driven by correlation, such as index tranches, but instead by *volatility*. More precisely, we deal with European-type optional payoffs where:

- the underlying is the *quoted* index spread[5]
- the payment nominal is the outstanding index notional. This condition ensures a natural match between the derivative and its hedge with the index, and simplifies computations.

Formally, we consider the payoffs at t of the form:

$$\varphi\left(S^q_{T,T_n}\right) N_T \tag{9}$$

for some function φ. When $\varphi(s) = (s-K)^+$, we get a caplet on the index spread[6]. By the way we also introduce the so-called *CDS-like* payoffs:

$$\psi\left(S_{T,T_n}\right) RBP_{T,T_n} \tag{10}$$

[5] Although the payoff can sometimes be expressed in terms of CDS-like spread, as is the case for the index option.

[6] Note that the standard credit index option does not fit within this family of payoffs, it will be addressed in §4.

We can always price these derivatives via an expectation under $\mathbf{Q}$:

$$\begin{aligned}\pi_t^{\varphi} &= ZC_{t,T}\mathbf{E}_t^{\mathbf{Q}}\left[\varphi\left(S_{T,T_n}^q\right)N_T\right] \\ \pi_t^{\psi} &= ZC_{t,T}\mathbf{E}_t^{\mathbf{Q}}\left[\psi\left(S_{T,T_n}\right)RBP_{T,T_n}\right]\end{aligned} \tag{11}$$

but this general approach does not account for the specificities of these payoffs. In the following, we first introduce models on the CDS-like spread, by analogy with the single-name case. These models are appealing due to their conceptual simplicity, but the CDS-like spread is not the natural underlying. Then we model directly the quoted spread, but struggle to define the forward versions of the spread and RBP.

3.1 Models on the CDS-like spread

We first extend to the *forward* case the definitions of the CDS-like spread and the RBP. We consider a *forward* maturity T, and we define the forward index RBP and the forward CDS-like index spread as the value at t of their payout at T :

$$\begin{aligned}RBP_{t,T,T_n} &\triangleq ZC_{t,T}\mathbf{E}_t^{\mathbf{Q}}\left[RBP_{T,T_n}\right] \\ S_{t,T,T_n}RBP_{t,T,T_n} &\triangleq ZC_{t,T}\mathbf{E}_t^{\mathbf{Q}}\left[S_{T,T_n}RBP_{T,T_n}\right]\end{aligned} \tag{12}$$

As in the single-name case, these can be computed using no-arbitrage conditions:

$$\begin{aligned}RBP_{t,T,T_n} &= RBP_{t,T_n} - RBP_{t,T} \\ S_{t,T,T_n}RBP_{t,T,T_n} &= S_{t,T_n}RBP_{t,T_n} - S_{t,T}RBP_{t,T}\end{aligned} \tag{13}$$

The process RBP_{t,T,T_n} appears as the natural numeraire for CDS-like payoffs (10), but it becomes becomes zero when all names default - this is the *Armageddon* event $\{N_T = 0\}$ introduced in [1]. Without loss of generality, we can focus on payoffs that are 0 upon Armageddon event[7], and for these we introduce the probability $\widetilde{\mathbf{Q}}$ defined by:

$$\left.\frac{d\widetilde{\mathbf{Q}}}{d\mathbf{Q}}\right|_t \triangleq \frac{RBP_{T,T_n}}{RBP_{t,T,T_n}}ZC_{t,T}$$

We call it the RBP probability, and definition (12) shows that it makes the forward CDS-like spread a martingale:

$$\mathbf{E}_t^{\widetilde{\mathbf{Q}}}\left[S_{T,T_n}\right] = S_{t,T,T_n}$$

[7] Otherwise split the payoff Π as a sum $\Pi.\mathbf{1}_{N_T>0} + \Pi.\mathbf{1}_{N_T=0}$, and price the second term independently, e.g. under the risk-neutral probability measure. So as to control the magnitude of this second term, and potentially neglect it, we need to estimate the Armageddon probability. Of course this probability will depend on the model chosen; we use a Gaussian copula with stochastic recovery (see [10]), and calibrate the correlation on the super senior tranche in the stressed markets of September 2008 . We then compute the probability for a 1-year maturity. For spreads around 500 bps, we get an Armageddon probability in the region of 0.1%, which should not affect the final price –unless Π is very large on this rare event.

As expected $\widetilde{\mathbf{Q}}$ is suited to CDS-like payoffs (10): they become 0 upon Aamgeddon event, and their $t-$price is given by a simple expectation:

$$\pi_t^{\psi} = RBP_{t,T,T_n} \mathbf{E}_t^{\widetilde{\mathbf{Q}}} \left[\psi \left(S_{T,T_n} \right) \right] \tag{14}$$

Assuming a log-normal diffusion for S_{t,T,T_n} under $\widetilde{\mathbf{Q}}$ will lead to a Black formula when ψ is a call function. For arbitrary functions ψ or more complex spread distributions, semi-closed formulae can be obtained via numerical integration.

Unfortunately the pricing of generic payoffs as in (9) exhibits convexity, cf. appendix C. When it comes to handling convexity, the models on the quoted spread that we introduce below will prove more natural.

3.2 Models on the quoted spread

3.2.1 Concepts and probabilistic tools

Index derivatives are naturally expressed in terms of the quoted spread, because it is readily available in the market. However, their pricing will involve the distribution of the quoted spread S^q_{T,T_n} at T and therefore will require a diffusion for some *forward* quoted spread S^q_{t,T,T_n}, which we now define with an eye on the definitions (12) of the CDS-like case:

- the forward flat RBP F_{t,T,T_n} is defined as the present value of the future flat RBP, with a notional term accounting for the losses occurred up to T :

$$F_{t,T,T_n}.N_t \triangleq ZC_{t,T} \mathbf{E}_t^{\mathbf{Q}} \left[F_{T,T_n} \left(S^q_{T,T_n} \right) .N_T \right] \tag{15}$$

- the forward upfront U_{t,T,T_n} is the discounted forward value of the upfront: $U_{t,T,T_n} \triangleq ZC_{t,T} \mathbf{E}_t^{\mathbf{Q}} \left[U_{T,T_n} \right]$. We rewrite this explicitly to introduce the unknown forward quoted spread S^q_{t,T,T_n} :

$$\left(S^q_{t,T,T_n} - S^c_{T_n} \right) F_{t,T,T_n}.N_t \triangleq ZC_{t,T} \mathbf{E}_t^{\mathbf{Q}} \left[\left(S^q_{T,T_n} - S^c_{T_n} \right) F_{T,T_n} \left(S^q_{T,T_n} \right) .N_T \right] \tag{16}$$

Definition (15) makes $\frac{F_{t,T,T_n} N_t}{ZC_{t,T}}$ a $\mathbf{Q}-$martingale. As in §3.1, we can focus on payoffs that are 0 upon Armageddon event, and define a new probability measure by:

$$\left. \frac{d\widetilde{\mathbf{Q}}^q}{d\mathbf{Q}} \right|_t \triangleq \frac{F_{T,T_n} \left(S^q_{T,T_n} \right) N_T}{F_{t,T,T_n} N_t} ZC_{t,T}$$

Rewriting (16) under this new probability immediately shows that the forward S^q_{t,T,T_n} becomes a martingale. Is this new probability useful ? For CDS-like payoffs (10), a change of probability had proved appropriate, leading to the simple pricing formula (14); but here, an expectation of (9) under $\widetilde{\mathbf{Q}}^q$ will introduce convexity, because the payoff considered does not contain the numeraire:

$$\pi_t^{\varphi} = F_{t,T,T_n} N_t \mathbf{E}_t^{\widetilde{\mathbf{Q}}^q} \left[\frac{\varphi\left(S_{T,T_n}^q\right)}{F_{T,T_n}\left(S_{T,T_n}^q\right)} \right] \tag{17}$$

At this stage we are left with an integration against the $\widetilde{\mathbf{Q}}^q-$distribution of the quoted spread upon exercise - typically this law will derive from the model chosen for the martingale S_{t,T,T_n}^q.

3.2.2 Joint computation of the forward RBP & spread

Unlike the CDS-like case, we can no longer interpret the forward RBP defined by (15) as a difference of two spot RBPs that could be read on the market. Instead, we take $\varphi \equiv 1$ in equations (11) and (17). We then equate these "prices" and get:

$$F_{t,T,T_n} = ZC_{t,T} \frac{\mathbf{E}_t^{\mathbf{Q}}\left[N_T\right]}{N_t \mathbf{E}_t^{\widetilde{\mathbf{Q}}^q}\left[1/F_{T,T_n}\left(S_{T,T_n}^q\right)\right]} \tag{18}$$

Now we rewrite the right-hand side of (16) by applying successively equations (7), (12) and (13), and get an equation where S_{t,T,T_n}^q is the only unknown, once we have computed F_{t,T,T_n} :

$$\left(S_{t,T,T_n}^q - S_{T_n}^c\right) F_{t,T,T_n}.N_t = \left(S_{t,T_n}^q - S_{T_n}^c\right) F_{t,T_n}\left(S_{t,T_n}^q\right) N_t - \left(S_{t,T} - S_{T_n}^c\right) RBP_{t,T} \tag{19}$$

In practice:

- The terms $\mathbf{E}_t^{\mathbf{Q}}\left[N_T\right]$ and $RBP_{t,T}$ require the bootstrapping of the notional decay rate η_t introduced in Appendix A, but only up to time T
- The denominator is obtained by numerical integration over the $\widetilde{\mathbf{Q}}^q-$distribution of S_{T,T_n}^q. Assume we pick a log-normal distribution, then it will be centered on the forward spread, which is unknown. Therefore the formula (18) must be plugged in (19), which now contains only the forward - explicitly, but also implicitly via the distribution of S_{T,T_n}^q.

Appendix D suggests two approximated frameworks that simplify the computations given in this section. It also contains a summary designed to facilitate the practical implementation of the exact and approximated frameworks.

4 Standard Index Options: how mischievous ?

4.1 Product description

In a payer option (call on protection, i.e. put on risk), the option holder has the right to buy protection on the index at a spread K called the strike, at some exercise date T. In practice, the exercise takes place at one of the index dates[8] T_i.

More precisely, in the case of a *payer* option, the trade confirmation states that, upon exercise at T, the option holder will:

- sell risk on a physical contract on the index, thus paying a spread $S^c_{T_n}$ to get the losses. By definition, the MtM of this contract is the upfront at $t = T$ as defined in equation (2).
- pay/receive an upfront payment

  $$u^* \triangleq \left(K - S^c_{T_n}\right) F_{T,T_n}(K),$$

  plus some accrual term that (imperfectly) nets with the index accrual of the physical index trade.
- receive the index losses L_T that occurred between the roll date T_0 and the exercise date T. Therefore an index option differs from a CDS option, as the latter knocks out if the underlying credit defaults: a CDS option provides protection against spread risk, but not default risk.

Finally, the payoff at exercise date T for a payer option is[9]:

$$\phi^P_T = \left[\left(S^q_{T,T_n} - S^c_{T_n}\right) F_{T,T_n}\left(S^q_{T,T_n}\right) N_T - u^* + L_T\right]^+ \tag{20}$$

Upon Armageddon event, the payoff degenerates into the constant $C \triangleq [L_{\max} - u^*]^+$, where $L_{\max} \triangleq \sum N^j \left(1 - R^j\right)$ is the maximum index loss. As suggested in §3.1, we split the payoff by writing:

$$\phi^P_T = \phi^P_T.\mathbf{1}_{N_T>0} + C.\mathbf{1}_{N_T=0}$$

The second term is simply a multiple of the Armageddon probability; in the following we will actually neglect this term, and apply equation (17) to get the option premium at t :

$$\pi_t\left(\phi^P_T\right) = F_{t,T,T_n}.N_t.\mathbf{E}^{\widetilde{\mathbf{Q}}^q}_t\left[\left(S^q_{T,T_n} - S^c_{T_n} + \frac{L_T - u^*}{F_{T,T_n}\left(S^q_{T,T_n}\right) N_T}\right)^+\right] \tag{21}$$

The conversion formula (8) allows to rewrite the payoff (20) in terms of CDS-like spreads. Therefore models on the CDS-like spread are just as relevant

[8] As of 2008, the liquid dates are T_1, T_2 and T_3, i.e. 3, 6 and 9 months from the roll date T_0

[9] We have carefully examined the legal documentation for a credit index option, and we are confident with the payoff. Nevertheless, some argue that the upfront cashflow u^* is actually paid on the risky notional N_T rather than the initial notional, assumed to be 1. In this case, the netting of the accruals is now perfect, and the results in the section remain valid provided the loss-adjusted spread introduced in §4.2.1 needs to be amended into: $S^*_{T,T_n} \triangleq S_{T,T_n} + \frac{L_T}{N_T \gamma_{T,T_n}}$.

for the credit index option, and all the results below will hold when using the following conversion table (see Appendix C for the definitions related to the CDS-like framework):

CDS-like	Quoted
$\widetilde{\mathbf{Q}}$	$\widetilde{\mathbf{Q}}^q$
RBP_{T,T_n}	$F_{T,T_n}\left(S^q_{T,T_n}\right).N_T$
RBP_{t,T,T_n}	$F_{t,T,T_n}.N_t$
γ_{T,T_n}	$F_{T,T_n}\left(S^q_{T,T_n}\right)$
S_{T,T_n}	S^q_{T,T_n}

In particular, the option premium given by (21) can be rewritten as:

$$\pi_t\left(\phi_T^P\right) = RBP_{t,T,T_n}\mathbf{E}_t^{\widetilde{\mathbf{Q}}}\left[\left(S_{T,T_n} - S^c_{T_n} + \frac{L_T - u^*}{\gamma_{T,T_n}N_T}\right)^+\right]$$

4.2 Pricing: How to incorporate the loss term ?

The payoff of credit index options involves *two correlated underlyings*: the index spread and the cumulated index loss. From a modelling point of view, their link is far from trivial; in fact, the spread S_{T,T_n} reflects an expectation of future losses *at the horizon* T_n, while the loss L_T represents the realized losses *at option maturity* T.

Replacing the loss term by its expectation is appealing (it easily leads to a Black formula) but has many shortcomings that are not discussed here. This section will focus on more sophisticated solutions: the first one introduces an auxiliary spread related to the realised loss, while the second one computes the loss conditionally on the spread.

4.2.1 Loss-Adjusted Spread

Several authors (see [5], [3], [1]) adjust the spread to get rid of the loss term in the payoff. Our loss-adjusted spread S^*_{T,T_n} is defined implicitly by:

$$\left(S^*_{T,T_n} - S_c\right)F_{T,T_n}\left(S^*_{T,T_n}\right) \triangleq \left(S^q_{T,T_n} - S^c_{T_n}\right)F_{T,T_n}\left(S^q_{T,T_n}\right).N_T + L_T$$

By construction this spread "absorbs" the loss and the notional reduction (the terms N_T and L_T are only on the right-hand side of the definition): upon default, the loss is translated into an add-on on the adjusted spread[10]. Clearly it is the natural underlying for the index option, since the premium given by (21) is now a simple formula, *without* loss term:

$$\pi_t\left(\phi_T^P\right) = ZC_{t,T}\mathbf{E}_t^{\mathbf{Q}}\left[\left(\left(S^*_{T,T_n} - S_c\right)F_{T,T_n}\left(S^*_{T,T_n}\right) - u^*\right)^+\right] \tag{22}$$

[10] In practice, after a default in the index, the market will quote the index spread with & without the defaulted name, and the former spread corresponds to our loss-adjusted spread.

At this stage, [3] postulates a constant spread drift under the risk-neutral measure, and calibrates it to market data. Instead, we adapt the approach introduced in §3.2 for models on the quoted spread, and define:

- the *adjusted* forward flat RBP:

$$F^*_{t,T,T_n} \triangleq ZC_{t,T}\mathbf{E}^{\mathbf{Q}}_t\left[F_{T,T_n}\left(S^*_{T,T_n}\right)\right]$$

- the *adjusted* forward spread, via:

$$\left(S^*_{t,T,T_n} - S^c_{T_n}\right)F^*_{t,T,T_n} \triangleq ZC_{t,T}\mathbf{E}^{\mathbf{Q}}_t\left[\left(S^*_{T,T_n} - S^c_{T_n}\right)F_{T,T_n}\left(S^*_{T,T_n}\right)\right]$$

We recommend to compute these exactly, as we did in the quoted case - given the loss term can generate a substantial spread adjustment, the approximations suggested in appendix D may be too inaccurate. More precisely, we associate a probability $\widetilde{\mathbf{Q}}^*$ to the numeraire F^*_{t,T,T_n}, and we easily get:

$$\begin{cases} \left(S^*_{t,T,T_n} - S^c_{T_n}\right)F^*_{t,T,T_n} = U_{t,T_n} - U_{t,T} + ZC_{t,T}\mathbf{E}^{\mathbf{Q}}_t\left[L_T\right] \\ F^*_{t,T,T_n} = \dfrac{ZC_{t,T}}{\mathbf{E}^{\widetilde{\mathbf{Q}}^*}_t\left[1/F_{T,T_n}\left(S^*_{T,T_n}\right)\right]} \end{cases} \tag{23}$$

As in the quoted spread case, we need CDS-like information to compute the expected loss and the upfront $U_{t,T}$; as for the upfront U_{t,T_n}, it can be computed with either CDS-like or quoted spreads. Finally the index option price writes:

$$\pi_t\left(\phi^P_T\right) = F^*_{t,T,T_n}\mathbf{E}^{\widetilde{\mathbf{Q}}^*}_t\left[\left(S^*_{T,T_n} - S_c - \frac{u^*}{F_{T,T_n}\left(S^*_{T,T_n}\right)}\right)^+\right],$$

and it only remains to perform a numerical integration over the chosen distribution for the adjusted spread.

4.2.2 Homogeneous Pool with Conditional Independence (HPCI)

In this approach we model the realized losses L_T *conditional* on the spread. This method relies on two main assumptions:

1. *Homogeneous pool*: The index constituents have the same default probability, weight and recovery. Let λ_t be the (common) intensity process and $\Lambda_{t,T} = \int_t^T \lambda_u du$. Conditionally on $\Lambda_{t,T}$, the (common) default probability is:

$$q \triangleq \widetilde{\mathbf{Q}}^q_t\left(\tau_j > T \middle| \Lambda_{t,T}\right) = e^{-\Lambda_{t,T}}$$

2. *Conditional independence*: The default events are independent conditionally on $\Lambda_{t,T}$

We actually condition (21) on $\Lambda_{t,T}$ (rather than the spread, but we show below that this is equivalent) and we are left with computing the following expectation:

$$\pi\left(\Lambda_{t,T}\right) \triangleq \mathbf{E}_t^{\widetilde{\mathbf{Q}}^q}\left[\left(S_{T,T_n}^q - S_{T_n}^c + \frac{L_T - u^*}{F_{T,T_n}\left(S_{T,T_n}^q\right) N_T}\right)^+ \middle| \Lambda_{t,T}\right] \tag{24}$$

The homogeneity assumption yields the obvious relationship:

$$N_T = 1 - \frac{L_T}{1-R},$$

where R is the common recovery value. Therefore, if we are able to convert this conditioning on $\Lambda_{t,T}$ into a conditioning on S_{T,T_n}^q, only L_T will be random in (24). For that purpose, we replace the integral by a basic trapeze approximation:

$$\Lambda_{t,T} \approx \frac{\lambda_t + \lambda_T}{2}\left(T-t\right)$$

Now, for a short time horizon ε, we also have the following approximations:

$$\lambda_t \approx \frac{S_{t,t+\varepsilon}^q}{1-R} \text{ and } \lambda_T \approx \frac{S_{T,T+\varepsilon}^q}{1-R}$$

The value of $S_{t,t+\varepsilon}^q$ can be read by extrapolation of the index spread curve as of t. So as to express the unknown $S_{T,T+\varepsilon}^q$ as a function of the known S_{T,T_N}^q we need a *further* assumption on the moves of the spread curve between time t (today) and time T :

1. Homothecy: $\frac{S_{T,T_n}^q}{S_{T,T+\varepsilon}^q} = \frac{S_{t,T,T_n}^q}{S_{t,T,T+\varepsilon}^q}$
2. Translation: $S_{T,T_n}^q - S_{T,T+\varepsilon}^q = S_{t,T,T_n}^q - S_{t,T,T+\varepsilon}^q$

In both cases, we can write $S_{T,T+\varepsilon}^q = h\left(S_{T,T_n}^q\right)$ for some function h, and we get the required link between $\Lambda_{t,T}$ and S_{T,T_n}^q :

$$\Lambda_{t,T} = \frac{T-t}{2\left(1-R\right)}\left[S_{t,t+\varepsilon}^q + h\left(S_{T,T_n}^q\right)\right]$$

Finally (24) can be rewritten as:

$$\pi\left(\Lambda_{t,T}\right) = \widetilde{\pi}\left(S_{T,T_n}^q\right)$$

with:

$$\begin{aligned}\widetilde{\pi}\left(s\right) &\triangleq \mathbf{E}_t^{\widetilde{\mathbf{Q}}^q}\left[\left(s - S_{T_n}^c + \frac{1}{F_{T,T_n}\left(s\right)}\frac{L_T - u^*}{1-\frac{L_T}{1-R}}\right)^+ \middle| S_{T,T_n}^q = s\right] \\ &= \mathbf{E}_t^{\widetilde{\mathbf{Q}}^q}\left[\Phi\left(L_T, s\right)^+ \middle| S_{T,T_n}^q = s\right]\end{aligned}$$

From there we only need L_T : given $S^q_{T,T_n} = s$, we know the common intensity $\Lambda_{t,T}$, and then the common default probability q. At this stage we use the conditional independence assumption to build the distribution of L_T by a plain recursion.

Unfortunately, the assumptions introduced so far will bias the index expected loss. Therefore we rescale the common default intensity $\Lambda_{t,T}$ with a constant factor to guarantee that the market-implied (discounted) expected loss is matched by the model. Within the model, this discounted expected loss reads:

$$F_{t,T,T_n}.\mathbf{E}_t^{\widetilde{\mathbf{Q}}^q}\left[\frac{1}{F_{T,T_n}\left(S^q_{T,T_N}\right)}\underbrace{\mathbf{E}_t^{\widetilde{\mathbf{Q}}^q}\left[\frac{L_T}{N_T}\middle| S^q_{T,T_N}\right]}_{\text{depends on scaling factor}}\right]$$

Implying the scaling factor, which generally lays in the interval $[80\%, 120\%]$, is actually not so expensive - computing an expected loss in the model can be done in a fraction of a second. Note that other methods can be used to compute the loss distribution, see [11].

5 Conclusion

We detailed two robust pricing frameworks for index spread derivatives. CDS-like spreads are obtained by bootstrapping, but can be handled by a straightforward translation of the single name approach. On the other hand, quoted spreads are observable on the market, but the practical implementation requires approximations.

Neither framework is sufficient to cope with the Credit Index Option, given its payoff incorporates the realized index losses up to exercise date, on top of the underlying spread. We proposed two solutions. The adjusted-spread approach is an *ad hoc* extension of our frameworks, whereas the HPCI models the joint behavior of the spread and the realised loss, and as such is more general.

All the methodologies introduced in this paper must be combined with a choice of dynamics for the spread chosen (quoted, CDS-like, or adjusted). The driver for such a choice is the fit to the options premiums observed on the market. Here we have only mentioned the lognormal distribution, but local spread volatilities, CEV dynamics, or Black-Karasinski are definitely worth exploring.

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A Bootstrapping of a CDS-like curve from a Quoted curve

This section describes the conversion of a standard (quoted) spread curve into a CDS-like spread curve. We start by rewriting the MtM at t of a credit index contract where we buy protection up to some maturity T_n at some contractual spread $S^c_{T_n}$:

$$U_{t,T_n} = \mathbf{E}^{\mathbf{Q}}_t \left[\int_t^{T_n} ZC_{t,s} dL_s \right] - S^c_{T_n} \sum_{t<T_i \leq T_n} ZC_{t,T_i} \mathbf{E}^{\mathbf{Q}}_t \left[\int_{T_{i-1}}^{T_i} N_s ds \right]$$

Here $\mathbf{E}^{\mathbf{Q}}_t$ denotes the risk-neutral expectation with the information available at t. Assuming homogeneous losses with a common recovery rate R, we get:

$$U_{t,T_n} = -(1-R) \int_t^{T_n} ZC_{t,u} \mathbf{E}^{\mathbf{Q}}_t [dN_u] - S^c_{T_n} \sum ZC_{t,T_i} \int_{T_{i-1}}^{T_i} \mathbf{E}^{\mathbf{Q}}_t [N_u] du \quad (25)$$

We further assume a deterministic exponential decay rate $\eta_t(\cdot)$ for (the expectation of) the index nominal:

$$\mathbf{E}^{\mathbf{Q}}_t [N_u] = e^{-\int_t^u \eta_t(v)dv} N_t \quad (26)$$

This algorithm requires an assumption on $\eta_t(\cdot)$, e.g. a stepwise function:

$$\eta_t(u) = \sum_k \eta_k \mathbf{1}_{\left\{T_{i_k} \leq u < T_{i_k}\right\}}$$

where the dates T_{i_k} usually corresponds to the index maturities where a spread is quoted. We then plug (26) in equation (25), and build the function $\eta_t(\cdot)$ by bootstrapping, i.e. we apply (25) for the successive quoted maturities.

Remark 5 *This is similar to the bootstrapping of a deterministic CDS default intensity from CDS quotations made of an upfront amount along with a fixed running spread*[11]*: the left-hand side of (25) corresponds to the upfront, as computed from the quoted spread using equation (2), while the right-hand side involves the contractual running spread* $S^c_{T_n}$.

Once $\eta_t(\cdot)$ is calibrated, we can compute:

$$RBP_{t,T_n} = \left(\sum_{t<T_i \leq T_n} ZC_{t,T_i} \int_{T_{i-1}}^{T_i} e^{-\int_t^u \eta(v)dv} du \right) N_t$$

and from there the CDS-like spread can be computed via the conversion formula (8)

Example 6 *As of May 1, 2008, the mid 5Y XO spread is 423 bps in quoted version, to be compared with 426 bps in running version. A complete set of results and comments is given in appendix B.*

[11] Usually 500 bps for single-names CDSs that trade upfront.

B Numerical comparison between Quoted and CDS-like spreads

B.1 Spot spreads

We use market data as of May 1st, 2008, and focus on the Series 9 of the iTraxx XO (Crossover) and iTraxx IG (Investment Grade, a.k.a. iTraxx Main). Given the XO curve is not so steep on that date, we also include a ficticious -though realistic- curve, dubbed iTraxx XO "Steep". So as to compute CDS-like spreads, note that we have assumed a linear interpolation between the quoted spreads, and a flat extrapolation. Finally, all spreads are expressed in bps.

- *IG:* As expected, there is no difference for the first maturity; this results from our flat extrapolation of quoted spreads when building CDS-like spreads. The other differences are small, yet visible: the steepness of the quoted spread curve is somewhat mitigated by the relatively low spread levels.

Spreads \ T_n	20-Jun-11	20-Jun-13	20-Jun-15	20-Jun-18
$S^c_{T_n}$	140	165	170	175
S^q_{t,T_n}	48	68	73	78
S_{t,T_n}	48	69	74	79
Difference	0	1	1	1

RBPs \ T_n	20-Jun-11	20-Jun-13	20-Jun-15	20-Jun-18
$F_{t,T_n}(S^q_{t,T_n})$	2.91	4.51	5.94	7.77
RBP_{t,T_n}	2.91	4.54	5.97	7.82
Relative diff.	0%	1%	1%	1%

- *XO:* The results below are crucial given that the 5Y XO is the usual underlying for credit index options. Nevertheless, as announced, the relatively flat XO curve does not allow to clearly differentiate the spreads:

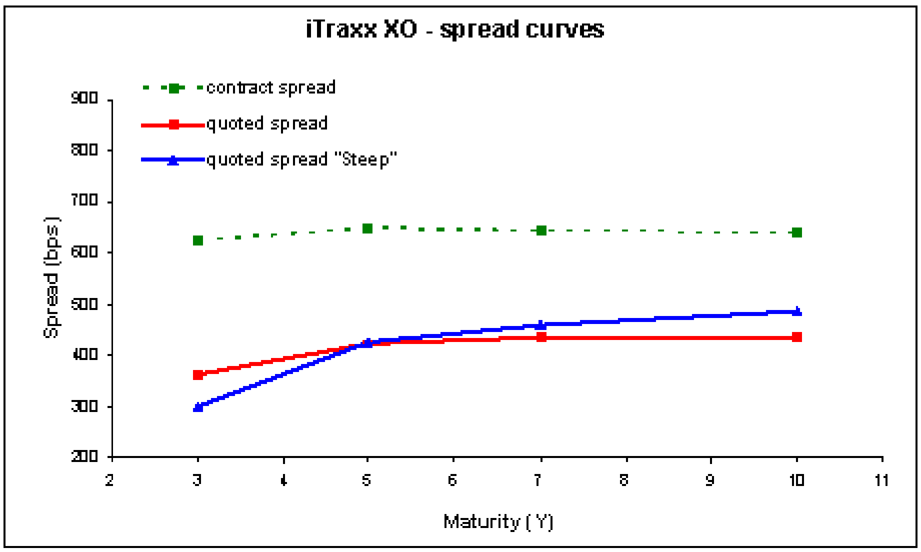
Spreads \ T_n	20-Jun-11	**20-Jun-13**	20-Jun-15	20-Jun-18
$S^c_{T_n}$	625	**650**	645	640
S^q_{t,T_n}	362	**423**	436	436
S_{t,T_n}	362	**426**	439	437
Difference	0	**3**	3	1

RBPs \ T_n	20-Jun-11	20-Jun-13	20-Jun-15	20-Jun-18
$F_{t,T_n}(S^q_{t,T_n})$	2.69	3.91	4.89	5.99
RBP_{t,T_n}	2.69	3.97	4.95	6.03
Relative diff.	0%	1%	1%	1%

- *XO "Steep":* The 10Y–3Y spread is 186 bps for our Steep curve, to be compared to 74 bps for the real curve. The differences are substantial, which confirms steepness as a key driver:

Spreads \ T_n	20-Jun-11	20-Jun-13	20-Jun-15	20-Jun-18
$S^c_{T_n}$	625	650	645	640
S^q_{t,T_n}	300	425	458	486
S_{t,T_n}	300	432	464	491
Difference	0	7	6	4

RBPs \ T_n	20-Jun-11	20-Jun-13	20-Jun-15	20-Jun-18
$F_{t,T_n}(S^q_{t,T_n})$	2.73	3.91	4.83	5.79
RBP_{t,T_n}	2.73	4.03	4.98	5.95
Relative diff.	0%	3%	3%	3%



B.2 Forward spreads

We focus on the index contracts that underly the liquid credit index options. As of $t = 1$ May 2008, these start at a date T which is either of 20 June 2008, 20 September 2008, and 20 December 2008; they all mature at 5Y, i.e. $T_n = 20$ June 2013. We include two additional start dates (20 June 2009 and 20 June 2010) to open the door to options with longer maturity.

Quoted and adjusted spreads and flat RBP are computed using the exact formulae - resp. (31) and (23) - assuming a volatility of 50%.

- *XO:* We observe that quoted and CDS-like spreads are close, which is line with what was observed in the previous section. The adjusted spread increases much faster than the others, due to its loss component.

Spreads \ T	20-Jun-08	20-Sep-08	20-Dec-08	20-Jun-09	20-Jun-10
S^q_{t,T,T_n}	425	430	435	447	485
S_{t,T,T_n}	428	433	438	450	487
S^*_{t,T,T_n}	439	471	504	582	806

RBPs \ T	20-Jun-08	20-Sep-08	20-Dec-08	20-Jun-09	20-Jun-10
F_{t,T,T_n}	3.78	3.53	3.30	2.85	2.02
RBP_{t,T,T_n}	3.83	3.58	3.35	2.89	2.04
F^*_{t,T,T_n}	3.79	3.56	3.35	2.93	2.12

- *XO "Steep":* As for the spot case, the steepness of the curve of quoted spreads is a key driver, as evidenced below:

Spreads \ T	20-Jun-08	20-Sep-08	20-Dec-08	20-Jun-09	20-Jun-10
S^q_{t,T,T_n}	430	439	450	475	552
S_{t,T,T_n}	436	445	455	479	554
S^*_{t,T,T_n}	441	473	507	587	820

RBPs \ T	20-Jun-08	20-Sep-08	20-Dec-08	20-Jun-09	20-Jun-10
F_{t,T,T_n}	3.78	3.54	3.30	2.86	2.03
RBP_{t,T,T_n}	3.88	3.64	3.40	2.94	2.07
F^*_{t,T,T_n}	3.79	3.56	3.34	2.92	2.12

C Applying CDS-like models to payoffs on the quoted spread

C.1 Convexity arising from the model/product mismatch

The $t-$price of a general derivative on the quoted index spread, with payoff as in (9), can be written as a risk-neutral expectation:

$$\pi^\varphi_t = ZC_{t,T}\mathbf{E}^{\mathbf{Q}}_t\left[\varphi\left(S^q_{T,T_n}\right)N_T\right]$$

Unfortunately, rewriting it with the RBP probability brings no obvious simplification for these derivatives:

$$\pi^\varphi_t = RBP_{t,T,T_n}\mathbf{E}^{\widetilde{\mathbf{Q}}}_t\left[\frac{\varphi\left(S^q_{T,T_n}\right)}{\gamma_{T,T_n}}\right] \tag{27}$$

where we need to introduce a *normalized index RBP*:

$$\gamma_{T,T_n} \triangleq \frac{RBP_{T,T_n}}{N_T}$$

We will also need auxiliary processes:

$$\gamma_{t,T,T_n} \triangleq \frac{RBP_{t,T,T_n}}{ZC_{t,T}\mathbf{E}_t^{\mathbf{Q}}\left[N_T\right]}$$

$$c_{t,T,T_n} \triangleq \frac{\gamma_{t,T,T_n}}{\gamma_{T,T_n}} - 1$$

where the expectation $\mathbf{E}_t^{\mathbf{Q}}\left[N_T\right]$ is given by (26).

From the definitions above, it is easy to prove that $1/\gamma_{t,T,T_n}$ is a $\widetilde{\mathbf{Q}}-$martingale:

$$\frac{1}{\gamma_{t,T,T_n}} = \mathbf{E}_t^{\widetilde{\mathbf{Q}}}\left[\frac{1}{\gamma_{T,T_n}}\right] \tag{28}$$

so that:

$$\mathbf{E}_t^{\widetilde{\mathbf{Q}}}\left[c_{t,T,T_n}\right] = 0$$

which leads us to rewrite (27) as a sum of a simple expectation and a convexity adjustment:

$$\pi_t^{\varphi} = ZC_{t,T}\mathbf{E}_t^{\mathbf{Q}}\left[N_T\right]\left\{\mathbf{E}_t^{\widetilde{\mathbf{Q}}}\left[\varphi\left(S^q_{T,T_n}\right)\right] + \underbrace{\mathbf{E}_t^{\widetilde{\mathbf{Q}}}\left[\varphi\left(S^q_{T,T_n}\right)c_{t,T,T_n}\right]}_{\text{convexity adjustment}}\right\} \tag{29}$$

This highlights two shortcomings when we use a framework on the CDS-like spread for derivatives on the quoted spread:

- The convexity adjustment c_{t,T,T_n} depends on the whole term-structure of the spread (via γ_{T,T_n}, which itself involves RBP_{T,T_n}); therefore we will attempt to write it as a function of the spread.
- Similarly, the quoted spread should also be expressed as a function of the CDS-like spread.

Both of these issues will be addressed below, at the cost of a few approximations.

C.2 Approximations of the convexity adjustment

Following [9], we approximate γ_{T,T_n}(and hence c_{t,T,T_n}) by a simple function $g\left(S_{T,T_n}\right)$ of the CDS-like spread. To make sure that the approximation properly degenerates in the case of deterministic spreads, we rescale the above and write instead:

$$\gamma_{T,T_n} \approx \hat{g}\left(S_{T,T_n}\right) \triangleq g\left(S_{T,T_n}\right)\frac{\gamma_{t,T,T_n}}{g\left(S_{t,T,T_n}\right)}$$

The convexity adjustment becomes:

$$c_{t,T,T_n} = \frac{g\left(S_{t,T,T_n}\right)}{g\left(S_{T,T_n}\right)} - 1$$

As expected, the adjustment becomes 0 when spreads are deterministic.

Different suggestions for the function g appear in the interest-rate litterature, see [9]. For our case, we believe that the Flat RBP introduced in §2.2 is a good proxy of the normalized index RBP, and finally we work with:

$$g\left(S_{T,T_n}\right) \triangleq F_{T,T_n}\left(S_{T,T_n}\right)$$

C.3 Quoted spread as a function of the CDS-like spread

A first approach assumes that the ratio of the spreads is the ratio of their forward values:

$$S^q_{T,T_n} \triangleq S_{T,T_n}\frac{S^q_{t,T,T_n}}{S_{t,T,T_n}}$$

where the *forward* quoted spread S^q_{t,T,T_n} is defined in §3.2. This solution is simple, but is less appropriate for wide/steep index spread curves.

Our second approach is more refined, but computationally intensive. From the conversion formula (8) between the two spreads, we get:

$$S^q_{T,T_n} = u^{-1}_{T,T_n}\left(\left(S_{T,T_n} - S^c_{T_n}\right)\gamma_{T,T_n}\right)$$

and where u^{-1} is the inverse function of u, that is $u \circ u^{-1}$ =Identity. Using the approximation introduced above for γ_{T,T_n}, we finally set:

$$S^q_{T,T_n} \triangleq u^{-1}_{T,T_n}\left(\left(S_{T,T_n} - S^c_{T_n}\right)\hat{g}\left(S_{T,T_n}\right)\right)$$

D Practical approximations in models on the quoted spread

D.1 Full-quoted framework

This simple framework uses only quoted spreads, and therefore avoids bootstrapping:

- the forward flat RBP is approximated as the difference of two *spot* flat RBPs, both computed at the (unknown) forward level:

$$F_{t,T,T_n} \approx F_{t,T_n}\left(S^q_{t,T,T_n}\right) - F_{t,T}\left(S^q_{t,T,T_n}\right)$$

- given the horizon T is short (typically a few months), the numerical results in B provide an empirical justification of the following approximations:

$$\begin{aligned} RBP_{t,T} &\approx F_{t,T}\left(S^q_{t,T}\right)N_t \\ S_{t,T} &\approx S^q_{t,T} \end{aligned}$$

This allows to rewrite (19) in terms of quoted spreads only:

$$\left(S^q_{t,T,T_n} - S^c_{T_n}\right)F_{t,T,T_n} = \left(S^q_{t,T_n} - S^c_{T_n}\right)F_{t,T_n}\left(S^q_{t,T_n}\right) - \left(S^q_{t,T} - S^c_{T_n}\right)F_{t,T}\left(S^q_{t,T}\right)$$

From there the quoted spread is easily implied.

D.2 Mixed framework

In the full-quoted framework above, the forward flat RBP is computed as the PV of 1bp cash-flows paid between T and T_n, with a *flat* risk given by our unknown forward quoted spread. Unfortunately this flatness assumption will bias the term N_T in definition 15... The mixed framework aims to better capture the short-term risk through the following rescaling:

$$F_{t,T,T_n} \approx \alpha_{t,T}.\left(F_{t,T_n}\left(S^q_{t,T,T_n}\right) - F_{t,T}\left(S^q_{t,T,T_n}\right)\right)$$

with:

$$\alpha_{t,T} \triangleq \frac{\mathbf{E}^{\mathbf{Q}}_t\left[N_T\right]}{\widetilde{\mathbf{E}}^{\mathbf{Q}}_t\left[N_T\right]} = \frac{e^{-\int_t^T \eta_t(u)du}}{e^{-\int_t^T \widetilde{\eta}_t(u)du}}$$

where $\widetilde{\mathbf{E}}^{\mathbf{Q}}_t$ and $\widetilde{\eta}_t$ are the versions of $\mathbf{E}^{\mathbf{Q}}_t$ and η_t corresponding to the case where the index spread curve is flat[12] at S^q_{t,T,T_n}. The short-term risk is now correct; when $T = T_m$ for some m, this is evidenced by the following

$$F_{t,T_m,T_n} = \sum_{i=m+1}^{n} \left(T_i - T_{i-1}\right) ZC_{t,T_i} e^{-\left(\int_t^{T_m} \eta_t(u)du + \int_{T_m}^{T_i} \hat{\eta}_t(u)du\right)}$$

This requires the bootstrapping of the notional decay rate η_t described in A - up to time T only, so the computational cost should remain reasonable. As for the spread, it is computed directly via (19) as for the exact case.

D.3 Summary

We end up with three possible set of definitions for the forward quoted spread and forward flat RBP:

- the *full-quoted* framework uses only quoted spreads:

$$\begin{cases} \left(S^q_{t,T,T_n} - S^c_{T_n}\right) F_{t,T,T_n} = \left(S^q_{t,T_n} - S^c_{T_n}\right) F_{t,T_n}\left(S^q_{t,T_n}\right) - \left(S^q_{t,T} - S^c_{T_n}\right) F_{t,T}\left(S^q_{t,T}\right) \\ F_{t,T,T_n} = F_{t,T_n}\left(S^q_{t,T,T_n}\right) - F_{t,T}\left(S^q_{t,T,T_n}\right) \end{cases}$$

- the *mixed* framework is more accurate but requires a bootstrapping:

$$\begin{cases} \left(S^q_{t,T,T_n} - S^c_{T_n}\right) F_{t,T,T_n} N_t = \left(S^q_{t,T_n} - S^c_{T_n}\right) F_{t,T_n}\left(S^q_{t,T_n}\right) N_t - \left(S_{t,T} - S^c_{T_n}\right) RBP_{t,T} \\ F_{t,T,T_n} = \left(F_{t,T_n}\left(S^q_{t,T,T_n}\right) - F_{t,T}\left(S^q_{t,T,T_n}\right)\right) \alpha_{t,T}\left(S^q_{t,T,T_n}\right) \end{cases} \quad (30)$$

[12] We have already mentioned that $\hat{\eta}_t(u) \approx \frac{S^q_{t,T,T_n}}{1-R}$.

- the *exact* framework relies on both bootstrapping and numerical integration:

$$\left\{\begin{array}{l} \left(S^q_{t,T,T_n} - S^c_{T_n}\right) F_{t,T,T_n} N_t = \left(S^q_{t,T_n} - S^c_{T_n}\right) F_{t,T_n}\left(S^q_{t,T_n}\right) N_t - \left(S_{t,T} - S^c_{T_n}\right) RBP_{t,T} \\ F_{t,T,T_n} = \frac{ZC_{t,T}\mathbf{E}^{\mathbf{Q}}_t[N_T]}{N_t \mathbf{E}^{\bar{\mathbf{Q}}^q}_t\left[1/F_{T,T_n}\left(S^q_{T,T_n}\right)\right]} \end{array}\right. \tag{31}$$

Remark 7 *For deterministic rates and spreads, the mixed and exact frameworks match. Indeed, the only randomness left in this case is the loss, impacting only N_t, and exactly reflected by $\alpha_{t,T}$.*

D.4 Numerical results

The approximations introduced in this appendix are fast and remain relatively accurate, but they both miss the convexity of the quoted spread: its forward should depend on its volatility. Here we provide numerical insight, in the case of a log-normal diffusion with 50% or 100% volatility for the forward quoted Crossover spread. As in appendix B, we take a market date $t = 1$ May 2008, and we focus on the 5y maturity: $T_n = 20$ June 2013. The results below show that the *mixed* approximation is an accurate proxy: the error always remains under 3bp for the spread (and 0.3 for the RBP). This is consistently better than the *full quoted* framework.

$S^q_{t,T,T_n} \setminus T$	20-Jun-08	20-Sep-08	20-Dec-08	20-Jun-09	20-Jun-10
full quoted	425	429	434	444	478
mixed	425	430	435	448	485
exact ($\sigma = 50\%$)	425	430	435	447	485
exact ($\sigma = 100\%$)	425	429	434	446	482

$F_{t,T,T_n} \setminus T$	20-Jun-08	20-Sep-08	20-Dec-08	20-Jun-09	20-Jun-10
full quoted	3.77	3.52	3.28	2.81	1.94
mixed	3.78	3.54	3.30	2.85	2.02
exact ($\sigma = 50\%$)	3.78	3.53	3.30	2.85	2.02
exact ($\sigma = 100\%$)	3.77	3.52	3.28	2.83	1.99